Moduli of K3 surfaces and mirror symmetry

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Moduli of K3 surfaces

K3: one complex family, dim $= 20$.

$H^2(\mathrm{K3}, \mathbb{Z}) = -E_8^{\oplus 2} \oplus H^{\oplus 3} = \tilde{L}$ a lattice, $\operatorname{rk} \tilde{L} = 2 \cdot 8 + 3 \cdot 2 = 22$.

$H^2(\mathrm{K3}, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$, dimensions $(1, 20, 1)$

$H^{2,0} = \mathbb{C} \cdot \alpha$ with α the holomorphic volume form.

Torelli theorem: $\mathbb{C}\alpha \subset H^2(\mathrm{K3}, \mathbb{C}) \simeq \tilde{L}_{\mathbb{C}}$ determines the Hodge structure and the isomorphism class of the K3.

Period domain: $\tilde{\Omega} = \left\{ \alpha \in \mathbb{P}(\tilde{L}_{\mathbb{C}}) \,\middle|\, \langle \alpha, \alpha \rangle = 0, \langle \alpha, \bar{\alpha} \rangle > 0 \right\}$

Moduli space of K3: $\{\mathrm{K3}\}/\mathrm{isom} = \tilde{\Omega} / \operatorname{Aut}(\tilde{L}, \langle . \, , . \rangle)$

Is non-Hausdorff...

Note: $\operatorname{Pic}(X) = \mathrm{NS}(\mathrm{X}) = H^{1,1} \cap H^2(X, \mathbb{Z}) = (\mathbb{R}\alpha + \mathbb{R}\bar{\alpha})^{\perp} \cap \tilde{L}$ varies: $\operatorname{rank} \operatorname{Pic}(X) \in \{0, \dots, 20\}$ (all cases occur).

Moduli of polarized K3 surfaces

Fix class of polarizing line bundle $\mathcal{L}$: $h = c_1(\mathcal{L}) \in \tilde{L}$.

Degree: $h^2 = 2g - 2$ is even, $g \geq 2$ the genus of a hyperplane section.

For any $g \geq 2$ we have a nice moduli stack of genus g polarized K3 surfaces:

Restricted lattice: $L = L(g) = -E_8^{\oplus 2} \oplus H^{\oplus 2} \oplus \langle 2 - 2g \rangle = h^\perp$

$\mathrm{sign}(L) = (2, 19)$.

Period domain: $\Omega = \left\{ \alpha \in \mathbb{P}(L_{\mathbb{C}}) \,\middle|\, \langle \alpha, \alpha \rangle = 0, \langle \alpha, \bar{\alpha} \rangle > 0 \right\}$

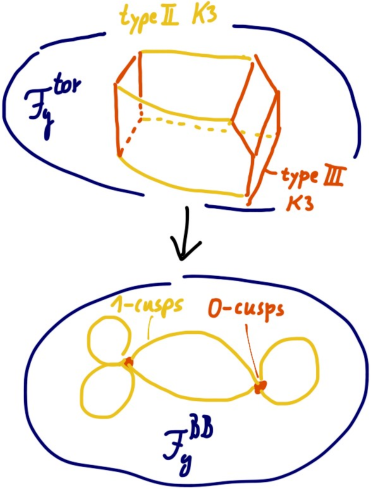
Moduli space for genus g: $\boxed{\mathscr{F}_g = \Omega / \operatorname{Aut}(L, \langle\, .,. \rangle)}$

This is now a Deligne-Mumford stack with an underlying quasi-projective variety.

For general $X \in \mathscr{F}_g$: $\mathrm{Pic}(X) = \mathbb{Z} \cdot h$.

Baily-Borel and toroidal compactification

Toroidal compactification: Is locally analytically toric, so adds divisors to $\mathscr{F}_g$. Depends on the choice of an infinite fan on each 0-cusp.



Baily-Borel compactification: A minimal compactification for any quotient of a Hermitian symmetric domain by an arithmetic group.

For $\mathscr{F}_g$: Adds points (0-cusps) and curves connecting the points (1-cusps)

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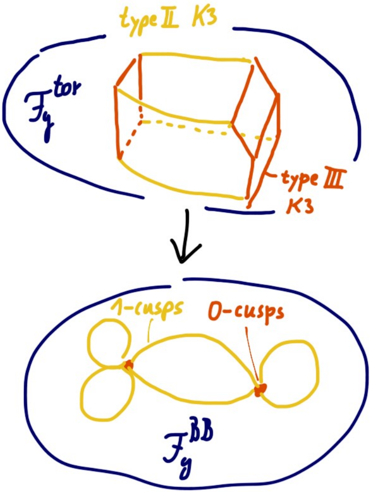
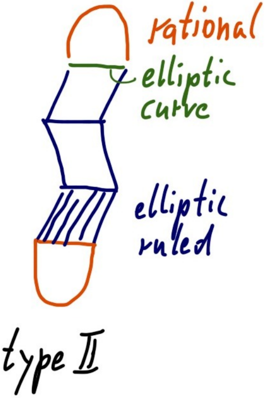
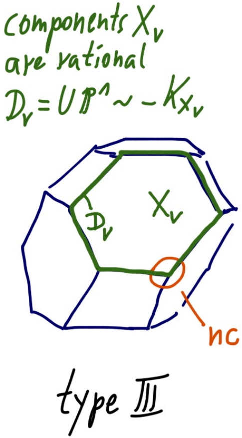


Degenerations of K3 surfaces

Two possibilities for central fibres in one-parameter (!) normal crossings degenerations $\mathfrak{X} \to D$ with $K_{\mathfrak{X}} = 0$ (Kulikov, Persson/Pinkham).

These are the respective fibres over 0- and 1-cusps in one-parameter normal crossing families.





The modular compactification problem

So far, there is no compactification of $\mathscr{F}_g$ known with an extension of the universal family of K3 surfaces.

D. Morrison (1993): Mirror symmetry may suggest natural choices of fans by Mori theory of the mirror.

Hacking/Keel: Use our smoothing algorithm (Gross/S. 2007) to also extend the family.

Theorem (Gross, Hacking, Keel, S.). There are toroidal compactifications of $\mathscr{F}_g$ supporting a universal family. (And these are all mutually compatible.)

Only minimal contributions from my side...

The mirror family

(Nikulin, Aspinwall, Voisin, Dolgachev)

Mirror family to $\mathscr{F}_g$: A one-dimensional modular family $\mathcal{Y}^* \to S^*$ of lattice polarized K3 surfaces: $\check{M} \subset H^2(\mathrm{K3}, \mathbb{Z})$.

$\check{M}$ depends on the choice[1] of a 0-cusp of $\mathscr{F}_g$.

Choice of 0-cusp: $f \in h^\perp \subset L$ (an SYZ Lagrangian fibre), and $s \in h^\perp$ with $f \cdot s = 1$, $s^2 = 0$ (a Lagrangian section of the SYZ fibration).

Mirror lattice: $\check{M} = \langle h, f \rangle^\perp / \langle f \rangle$ (rank 19, signature (1,18)).

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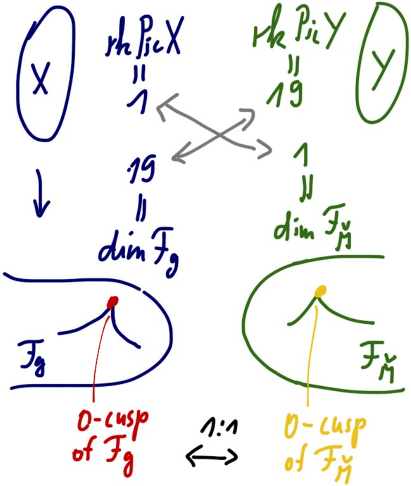
[1] If $2g - 2 = n \cdot k^2$ with n square-free, the number of 0-cusps is $[\frac{k}{2}] + 1$.

Complex versus Kähler moduli

$\dim \mathscr{F}_g = 19$
$\mathrm{rk}(\mathrm{Pic}(X_{\mathrm{gen}})) = 1.$

$\dim \mathscr{F}_{\check{M}} = \dim S^* = 1$
$\mathrm{rk}(\mathrm{Pic}(Y_{\mathrm{gen}})) = 19.$

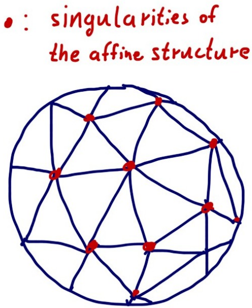


Local extensions of $\mathscr{F}_g$

Each nc extension $\mathcal{Y}$ of $\mathcal{Y}^*$ over the 0-cusp of S^* provides a partial compactification of $\mathscr{F}_g$ via mirror symmetry as follows.

Steps:

- The dual intersection complex of $Y_0 \subset \mathcal{Y}$ is a triangulated 2-sphere. The intersection numbers of the $\mathbb{P}^1$ in $(Y_0)_{\mathrm{sing}}$ determine a $\mathbb{Z}$-affine structure away from the vertices.



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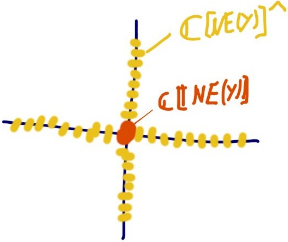
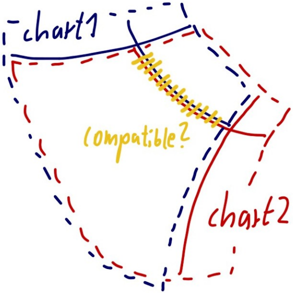
Steps:

- On the $\mathscr{F}_g$-side, mirror symmetry suggests to start from $X_0 = \bigcup \mathbb{P}^2$, one copy of $\mathbb{P}^2$ polarized by $\mathcal{O}(1)$ for each triangle.
- The irreducible components of Y_0 are log Calabi-Yau surfaces. Our smoothing algorithm then provides a smoothing of X_0 over $\operatorname{Spec} \mathbb{C}[\![\mathrm{NE}(\mathcal{Y})]\!]$. Here Gromov-Witten theory on the irreducible components of Y_0 replaces the KS scattering algorithm (Gross/Hacking/Keel).
- Since $\mathrm{rk}(\mathrm{NE}(\mathcal{Y})) = 19 + g$, the constructed family has g surplus dimensions. In fact, a g-dimensional torus acts on the construction and we can take a distinguished slice to cut down to 19 dimensions.

Extending and gluing the families

Local extension: The family extends locally to the completion of the toric type III locus in any toric modification of Spec $\mathbb{C}[\mathrm{NE}(\mathcal{Y})]$.





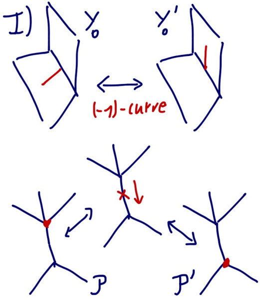
Gluing: The extension $\mathcal{Y}$ of $\mathcal{Y}^*$ can be changed by birational modifications on the central fibre.

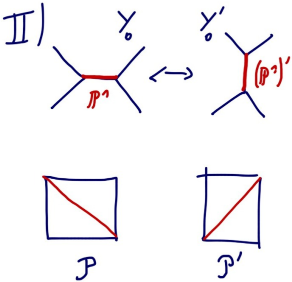
This leads to changes of the triangulation and different wall structures.

Need to glue the different partial compactifications of $\mathscr{F}_g$.

The importance of the central fibre

For the gluing it is important to control the effect of the standard birational transforms of $\mathcal{Y}$ on the wall structure.





Important: We need to work with the various $\mathrm{NE}(\mathcal{Y})$ rather than with the single $\mathrm{NE}(\mathcal{Y}^*_{\mathrm{gen}})$ to use mirror symmetry for the compactification.

The non-uniqueness of and control over the central fibre of $\mathcal{Y}$ is crucial!

Compactifying $\mathscr{F}_g$

Still need to attach our formal family to $\mathscr{F}_g$!

This amounts to computing the periods $\int_\beta \alpha$ as functions on $\operatorname{Spec} \mathbb{C}[\![\mathrm{NE}(\mathcal{Y})]\!]$.

By a result of Moret-Bailly it suffices to do this at any point of $\mathscr{F}_g^{\mathrm{tor}} \setminus \mathscr{F}_g$.

May thus reduce to a GS-like situation (purely toric models near 0-strata of X_0).

Strategy:

- Compute $H^2(X, \mathbb{Z})$ in terms of affine geometry and show the period integrals are monomial in $\mathbb{C}[\![\mathrm{NE}(\mathcal{Y})]\!]$.
- Identify $\check{M} = \operatorname{Pic}(\mathcal{Y}_{\mathrm{gen}})$.

The case with simple singularities

We now assume $\mathcal{Y}$ has a small contraction $\mathcal{Y} \to \overline{\mathcal{Y}}$ that is a toric degeneration as in GS (in particular, $\overline{Y}_0 = \bigcup$ toric varieties). The mirror family $\overline{\mathcal{X}} \to \overline{T}$ for $\overline{\mathcal{Y}}$ then covers an open subset of the mirror $\mathcal{X} \to T$ of $\mathcal{Y}$.

- The dual intersection complex B of $\overline{Y}_0$ is a triangulated $\mathbb{Z}$-affine manifold with 24 focus-focus singularities on the edges.

  Λ: the sheaf of $\mathbb{Z}$-tangent vectors, $i : B \setminus \{24 \text{ pts}\} \hookrightarrow B$.
- Vertices: $g+1$ (=number of irreducible components of $\overline{Y}_0$)
- Triangles: $2g-2$ (= t-invariant of Friedman/Scattone from monodromy of $\mathcal{Y} \to S$ = number of triple points)
- $H^2_{\lim}(\overline{\mathcal{Y}}, \mathbb{Z}) = H^0(B, \bigwedge^2(i_*\Lambda^*)) \oplus H^1(B, i_*\Lambda^*) \oplus H^2(B, \mathbb{Z})$
- The radiance obstruction $c_B \in H^1(B, i_*\Lambda)$ of B is the residue of the Gauss-Manin connection of $\mathcal{Y}^* \to S^*$ [GS II, 2007].

$$c_B^2 = (t - \text{invt}) = 2g - 2.$$

Essential tool: Period integrals

Theorem. (Ruddat/S.'14)

1) Period integrals can be computed and are monomial for any family β of 2-cycles on $\mathcal{X}$ constructed from a tropical 1-cycle $\beta_{\mathrm{trop}} \in H_1(B \setminus \Delta, \Lambda)$.

2) Tropical 1-cycles generate $H_1(B, i_*\Lambda)$.

3) The exponent of monomial of period integrals is determined by the canonical pairing

$$\langle . \, , . \rangle : H_1(B, i_*\Lambda) \otimes H^1(B, i_*\Lambda^*) \xrightarrow{\mathrm{Tr}\circ\cap} \mathbb{Z}.$$

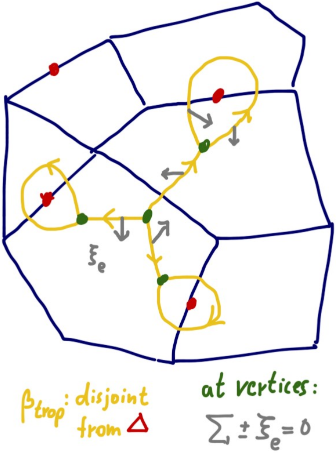
Here, $H^1(B, i_*\Lambda^*)$ carries monomial deformation parameters.

4) The fibrewise intersection product of the family of 2-cycles is compatible with the pairing

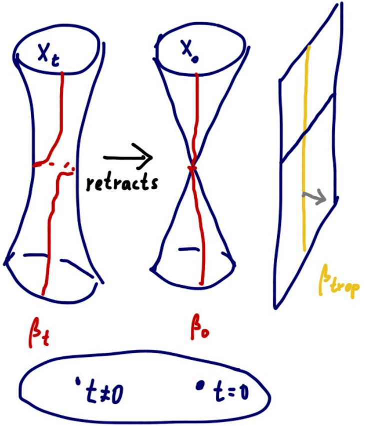
$$H_1(B, i_*\Lambda) \otimes H_1(B, i_*\Lambda) \longrightarrow H_0(B, \textstyle\bigwedge^2 i_*\Lambda) = \mathbb{Z}. \qquad \text{(IP)}$$

Cycles for the period integrals

tropical 1-cycle on B



associated family of 2-cycles in $\mathcal{X}$



$\mathrm{Pic}(\mathcal{Y}_{\mathbf{gen}})$

We may also compute period integrals for $\mathcal{Y} \to S$ by going over to the Legendre-dual and running GS'07:

$$(B, \mathscr{P}, \varphi) \longleftrightarrow (\check{B}, \check{\mathscr{P}}, \check{\varphi}).$$

Here $\mathscr{P}$ (the triangulation) and $\check{\mathscr{P}}$ are dual cell decompositions, $B = \check{B}$, but $\Lambda_{\check{B}} = \Lambda_B^*$.

$\mathrm{Pic}(\mathcal{Y}_{\mathbf{gen}})$

The family of 2-cycles β in $\mathcal{Y}^*$ for $\beta_{\mathrm{trop}} \in H_1(\check{B}, \check{i}_* \Lambda_{\check{B}})$ is Poincaré-dual to $H^{1,1}$ in the $\mathcal{Y}$-family iff

$$\int_\beta \alpha = 0.$$

For $\mathcal{Y} \to S$, the monomial deformation class is

$$c_1(\check{\varphi}\rangle \in H^1(\check{B}, \check{i}_* \Lambda^*_{\check{B}}).$$

So β is Poincaré-dual to a $(1,1)$-class in all $\mathcal{Y}$-fibres iff

$$\langle \beta_{\mathrm{trop}}, c_1(\check{\varphi}) \rangle = 0.$$

Thus: $$\boxed{\mathrm{Pic}(\mathcal{Y}_{\mathrm{gen}}) \overset{\mathrm{PD}}{=} c_1(\check{\varphi})^{\perp} \subset H_1(\check{B}, i_* \Lambda_{\check{B}})}$$

Corollary. $\mathcal{Y} \to S$ constructed from GS'07 also extends. $\mathcal{Y}^* \to S^*$
This extension is a toric degeneration, not an nc degeneration.

$\check{M}$ in affine geometry

Since $\mathcal{Y}$ is $\check{M}$-polarized, we know abstractly

$$\mathrm{Pic}(\mathcal{Y}_{\mathrm{gen}}) \simeq \check{M} = \langle h, f\rangle^{\perp}/\langle f\rangle.$$

But we need a canonical identification to use MoriFan($\mathcal{Y}$) for the toroidal compactification of $\mathscr{F}_g$!

The 2-cycle construction gives an embedding

$$\mathbb{Z}^{20} \simeq H_1(B, i_*\Lambda) \longrightarrow H_2(X_t, \mathbb{Z})/\langle f\rangle \simeq \mathbb{Z}^{21}.$$

In affine geometry, the polarizing class h on $\mathcal{X}$ is the radiance obstruction c_B of B:

$$h = c_B \in H^1(B, i_*\Lambda).$$

Lemma. For β_{trop} a tropical 1-cycle: $\int_\beta h = \langle \beta_{\mathrm{trop}}, h\rangle.$

$$\boxed{\check{M} = c_B^{\perp} \subset H^1(B, i_*\Lambda)}$$

Identification of (co-)homologies of both sides

Quite generally, there is also a canonical isomorphism of polyhedral cohomologies (Ruddat/S.'14):

$$H_i(B, i_*\Lambda) \simeq H^{\dim B - i}(B, i_*\Lambda) \quad (*)$$

We thus have the sequence of canonical isomorphisms:

$$\begin{aligned} H_1(\check{B}, \check{i}_*\Lambda_{\check{B}}) &\overset{\mathrm{Tr}\circ\cap}{=} H^1(\check{B}, \check{i}_*\Lambda^*_{\check{B}})^* \overset{\mathrm{LD}}{=} H^1(B, i_*\Lambda)^* \\ &\overset{(*)}{=} H_1(B, i_*\Lambda)^* \overset{\mathrm{IP}}{=} H_1(B, i_*\Lambda). \end{aligned}$$

Legendre-duality: $H^1(\check{B}, \check{i}_*\Lambda^*_{\check{B}}) \ni c_1(\check{\varphi}) \overset{\mathrm{LD}}{\longleftrightarrow} c_B \in H^1(B, i_*\Lambda)$

$$\Longrightarrow \quad \boxed{\check{M} = c_B^\perp = c_1(\check{\varphi})^\perp = \mathrm{Pic}(\mathcal{Y}_{\mathrm{gen}})}$$

Conclusion

In this application of mirror symmetry, it was both important:

- to control the birational **non-uniqueness of the central fibre** in patching local deformations and
- to relate the **integral** affine geometry of mirror symmetry to integral cohomology, monodromy actions and period integrals.

  This was possible by reduction to a situation with simple singularities on both sides.